Int. J. Heat Mass Transfer. Vol. 15, pp. 1171–1174. Pergamon Press 1972. Printed in Great Britain

SHORTER COMMUNICATIONS

THE USE OF GREEN'S FUNCTIONS FOR SOLVING MELTING OR SOLIDIFICATION PROBLEMS IN THE CYLINDRICAL COORDINATE SYSTEM

Y. K. CHUANG and J. SZEKELY

Department of Chemical Engineering and Center for Process Metallurgy, State University of New York at Buffalo, Buffalo, New York 14214, U.S.A.

(*Received* 30 *July* 1971 *and in revised form* 17 *November* 1971)

NOMENCLATURE

a^2,	thermal diffusivity;
c,	value defined in equation (10);
C_B,	carbon concentration in the bulk, in weight fraction;
${}_2F_1$	Gauss' hypergeometric series;
$G(r,t\|\xi,\tau)$,	Green's function, defined in equation (7);
h,	heat transfer coefficient;
h_D	mass transfer coefficient;
$I_0(z)$,	modified Bessel function of the first kind with order zero and argument z;
K,	thermal conductivity;
P_k,	partition point of the original cylinder radius R_1;
R_k,	melting front at t_k;
$R(t)$,	position of the melting front;
S_k,	slope of the chord connecting R_k and R_{k+1};
t_k,	time steps with $t_1 = 0$, etc.;
T, T_0,	solid temperatures at time t and time zero;
$T_B(t)$,	bulk temperature of the liquid bath;
$T_{mp}(t)$,	melting temperature;
α,	small constant value;
ΔH,	latent heat of melting;
ρ,	density;
τ,	time of occurrence of instantaneous source;
ξ,	position of instantaneous source.

1. INTRODUCTION

IN A PREVIOUS paper we described an analytical cum computational technique, using Green's functions for the solution of melting or solidification problems [1]. It was shown that the procedure outlined was attractive from the computational viewpoint, especially for systems involving non-linear boundary conditions or more than one dependent variable.

In this earlier paper the formulation was developed for one dimension in the rectangular coordinate system; the purpose of this communication is to show how these considerations may be applied to the melting or solidification of cylinders.

This approach may be of some practical interest because moving boundary problems in the cylindrical coordinate system do pose computational difficulties, although a useful approach has been recently suggested to these problems by Lederman and Boley [2].

2. FORMULATION

The application of the technique will be illustrated through the development of the equations and numerical solutions for the melting (and initial solidification) of an iron rod immersed into an iron–carbon melt. As discussed in the previous publication [1], under these conditions the melting process (and the initial build-up of a solid crust) is complicated by the fact that the diffusion of carbon will cause the effective melting point to be time dependent.

Let us consider a circular cylinder with an initial radius, R_1, and initial temperature T_0. At time, $t = 0$, the cylinder is brought into contact with an agitated liquid bath having a temperature $T_B(t)$, which is higher than the melting point of the cylinder $T_{mp}(t)$. On the assumptions of constant physical properties and purely radial heat transfer within the cylinder, the governing equation is readily written as

$$\frac{\partial T}{\partial t} = a^2\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right] \tag{1}$$

with boundary conditions

$$T = T_0, \quad t = 0 \tag{2}$$

$$T = T_{mp}(t), \quad r = R(t) \tag{3}$$

$$K\frac{\partial T}{\partial r}\bigg|_{R(t)} = h[T_B(t) - T_{mp}(t)] + \rho\Delta H\frac{\mathrm{d}R}{\mathrm{d}t} \tag{4}$$

and

$$\frac{\partial T}{\partial r} = 0, \quad r = 0 \tag{5}$$

where $R(t)$ is the moving boundary, h is the heat transfer coefficient, $\bar{K}$ is the thermal conductivity, a^2 is the thermal diffusivity, and ΔH is the latent heat of melting. By introducing the fundamental Green's function $G(r,t|\xi,\tau)$ and following the standard manipulations as described elsewhere [3,4], we obtain the temperature distribution in terms of surface boundary conditions in the following integral form:

$$T(r,t) = 2\pi \int_0^{R_1} T_0 \xi G|_{\tau=0}\, d\xi + 2\pi \int_{R_1}^{R(t)} T_{mp} \xi G|_{R(\tau)}\, d\xi + 2\pi a^2 \int_0^t \xi \left[G \frac{\partial T}{\partial \xi}\bigg|_{R(\tau)} - T_{mp} \frac{\partial G}{\partial \xi}\bigg|_{R(\tau)} \right] d\tau \quad (6)$$

where

$$G = G(r,t|\xi,\tau) = \frac{1}{4\pi a^2 (t-\tau)} \left\{ \exp\left[(-r^2 + \xi^2)/4a^2(t-\tau)\right] I_0\left(\frac{r\xi}{2a^2(t-\tau)}\right) \right\} \quad (7)$$

Here $I_0(z)$ is the modified Bessel function of the first kind with order zero and argument z. It is required that the temperature at the interface is at the melting point T_{mp}, which leaves the moving boundary $R(t)$ as the only unknown in the equation and therefore $R(t)$ may be sought by means of a simple numerical search. It is noted that in order to obtain the expression for the temperature at the interface, contour integration is necessary to perform the integration of equation (6) when $\tau \to t$ and $\xi \to r$ simultaneously. However, the resulting equation will exist only in the sense of a Cauchy principle value integral, which prevents the numerical integration from achieving desired degree of accuracy. This difficulty can be overcome by observing that the temperature, at a point infinitesimally close to the interface inside the solid, has almost the same value as the interface temperature T_{mp}. Thus on substituting $T = T_{mp}$ in equation (6), at such a point, the transcendental equation is ready for numerical search.

Except in very simple cases, numerical integration has to be used to evaluate the integral terms. By assuming a linear advancement of the interface and small change of T_{mp} and T_B in a short duration, equation (6) can be approximated by the following expression:

$$T_{mp} \cong \sum_{k=1}^{N} T_0(P_k) \int_{P_k}^{P_{k+1}} \xi G|_{\tau=0}\, d\xi + 2\pi \sum_{k=1}^{n} T_{mp}(t_k) \int_{R_k}^{R_{k+1}} \xi G|_{\tau=R(\tau)}\, d\xi + 2\pi a^2 \sum_{k=1}^{n} \left[\frac{h}{\bar{K}} (T_B(t_k) - T_{mp}(t_k)) + \frac{\rho \Delta H}{\bar{K}} \frac{dR}{dt} \right] \times \int_{t_k}^{t_{k+1}} \xi G\bigg|_{R(\tau)} d\tau - 2\pi a^2 \sum_{k=1}^{n} T_{mp}(t_k) \int_{t_k}^{t_{k+1}} \xi \frac{\partial G}{\partial \xi}\bigg|_{R(\tau)} d\tau. \quad (8)$$

The movement of the interface during solidification and melting is shown schematically in Fig. 1. The moving front

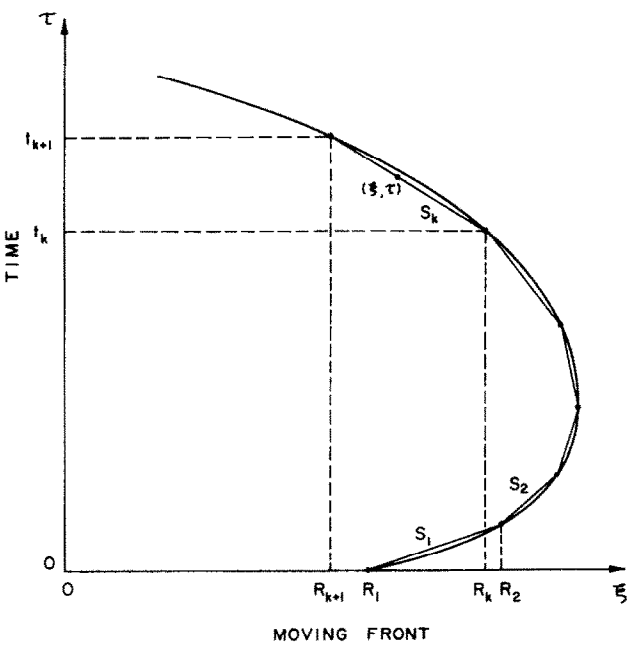

FIG. 1. Calculation of the movement of the interface.

is approximated by a series of cords with slopes S_k defined as

$$S_k = \frac{\xi - R_{k+1}}{t_{k+1} - \tau}, \qquad k = 0,1,\ldots,n \tag{9}$$

where

$$t_{n+1} = t$$

$$R_{n+1} = R(t).$$

Starting with the search on R_2 for $n = 2$, the value thus obtained is then used for the search on R_3 and so forth. By noticing that the maximum melting rate can be determined from equation (4) by setting $\partial T/\partial r = 0$, only a few iterations will be required for each search on R_k. For solidification, dR/dt at the current step becomes the upper bound of that of the subsequent time step. The numerical calculation is simplified further by using asymptotic expansion of $I_0(z)$ when the argument z becomes large [5]. Numerical comparison shows that the error caused by taking the leading term of the expansion is less than 0·64 per cent if the argument is larger than 20. Under these circumstances, some terms in equation (8) can be integrated analytically. Thus, for small value of α, the second term is integrated through the substitution of equation (9) to obtain

$$2\pi T_{mp} \int_{R+\alpha}^{R(t)} \xi G|_{\tau = R(\tau)}\, d\xi$$

$$\cong \frac{T_{mp}}{2a^2} \int_{R+\alpha}^{R(t)} \sqrt{\left(\frac{\xi}{t-\tau}\right)} \exp\left[-(R-\xi)^2/4a^2(t-\tau)\right] d\xi$$

$$= -T_{mp} \operatorname{erf}\left(\frac{\sqrt{c\alpha}}{c}\right) \tag{10}$$

where

$$c = 4a^2/S_k.$$

Similarly, the other terms become

$$2\pi a^2 \int_{t-\alpha}^{t} \xi G\Big|_{R(\tau)} \cong \frac{a^2}{S_n} \operatorname{erf}\left(\frac{S_n\sqrt{\alpha}}{2a}\right) \tag{11}$$

$$2\pi a^2 \int_{t-\alpha}^{t} \xi \frac{\partial G}{\partial \xi}\Big|_{R(\tau)} \cong \frac{1}{2}\left[1{\cdot}0 + \operatorname{erf}\left(\frac{S_n\sqrt{\alpha}}{2a}\right)\right]. \tag{12}$$

The first integration becomes exact when T_0 is a constant, thus

$$2\pi \int_0^{R_1} T_0 G\, \xi\Big|_{\tau=0}\, d\xi$$

$$= \frac{T_0 R_1^2}{4a^2 t} \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{r^2}{4a^2 t}\right)^m {}_2F_1\left(-m, -m; 2; \frac{R_1^2}{r^2}\right),$$

$$t > 0 \tag{13}$$

where ${}_2F_1[-m, -m; 2; (R_1^2/r^2)]$ is Gauss' hypergeometric series [5]. Equation (13) converges rapidly for large values of t and can be conveniently used instead of direct numerical integration.

A computed numerical example is given in the following:

3. COMPUTED NUMERICAL EXAMPLE

Let us consider the melting of a pure iron rod, having an initial radius of 0·52 in. and an initial temperature of 70°F, which is immersed into an iron–carbon melt, at time = 0.

Let the carbon concentration in the melt be 2·4 wt%, and let the temperature of the melt be given by the following expression:

$$T_B(t) = 2515{\cdot}0 - 0{\cdot}917\, t. \tag{14}$$

where t is in seconds

Because of the carbon diffusion, the melting point is changing according to the following expression [3]:

$$T_{mp} = 2800{\cdot}0 - 1{\cdot}5 \,.\, 10^4 \,.\, \frac{h_D \,.\, C_B}{h_D + |S_k|}. \tag{15}$$

Other data required are

$$h = 2000\,\text{Btu ft}^2/\text{h°F}$$

$$h_D = 0{\cdot}0152\,\text{in}^2/\text{s}$$

$$a^2 = 0{\cdot}018\,\text{in}^2/\text{s}$$

$$\Delta H = 110\,\text{Btu/lb}$$

$$\bar{K} = 20\,\text{Btu hft°F}$$

$$\rho = 445\,\text{lb/ft}^3.$$

The mass transfer coefficient h_D is related to the heat transfer coefficient h through the analogy between heat and mass transfer.

Equation (8) is immediately applicable here and the computed results are shown in Fig. 2. The actual computer time was about 50 s on the CDC 6400 computer of the State University of New York at Buffalo, most of this time was consumed by the calculations at the initial stage of the solidification, at the transition from solidification to melting, and at the point where the solidified crust disappears and the melting becomes diffusion controlled. The abrupt change in the slope of $t = 37{\cdot}5$ s is attributed to the difference in composition and hence the physical properties between the solidified crust and the original pure iron.

4. CONCLUSIONS

The application of the fundamental Green's Function to the moving boundary problems has converted the nonlinear differential equations into a single transcendental

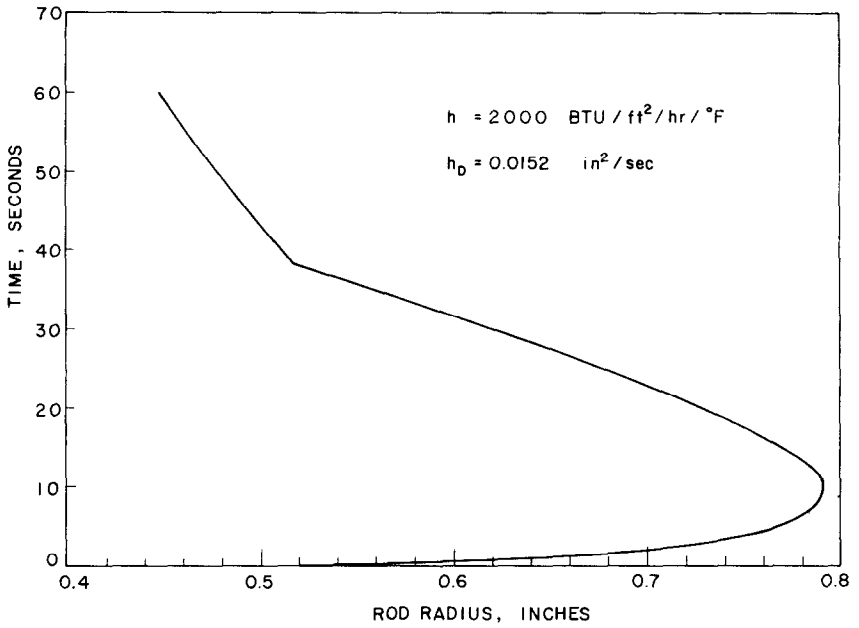

FIG. 2. Computed results, showing the rod radius as a function of time.

equation which may be solved by simple numerical calculations. The computational effort is greatly reduced because the calculation of the temperature distribution within the domain is unnecessary. The flexibility of this technique in tackling a rather complicated problem such as melting and solidification in multicomponent systems suggests this method should be very promising in treating a wide variety of moving boundary problems. For rectangular, cylindrical, and spherical coordinates, the fundamental solutions are available and their properties have been well studied. We may add that the unsteady state heat conduction equations is rectangular and spherical coordinates are mutually convertible through simple variable transformation. It follows, the moving boundary problem in spheri- coordinates, will follow our previous study [1] closely. It is noted that this technique can be applied to problems of higher dimensions, but at the expense of increased demand on computer time.

REFERENCES

1. Y. K. CHUANG and J. SZEKELY, On the use of Green's Function for solving melting or solidification problems, *Int. J. Heat Mass Transfer* **14**, 1285–1294 (1971).
2. J. M. LEDERMAN and B. A. BOLEY, Axisymmetric melting or solidification of circular cylinders, *Int. J. Heat Mass Transfer* **13**, 413–427 (1970).
3. Y. K. CHUAN , The melting and dissolution of a solid in a liquid with a strong exothermic heat of solution, Ph.D. Thesis, State University of New York at Buffalo (1971).
4. P. M. MORSE and H. FESHBACK, *Methods of Theoretical Physics*, Vol. 1. McGraw-Hill, New York (1961).
5. M. ABRAMOWITZ and I. A. STEGUN, *Handbook of Mathematical Functions*. Dover, New York (1965).